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Nonlinear boundary value problems of fractional differential systems ☆

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ABSTRACT

In this paper, we consider the existence of generalized solutions for fractional differential systems with nonlinear boundary value conditions. We first establish a new comparison theorem. By applying the monotone iterative technique and the method of lower and upper generalized solutions, we obtain sufficient conditions for existence of extremal generalized solutions.

1. Introduction

Differential equations with fractional order are a generalization of ordinary differential equations to non-integer order. This generalization is not a mere mathematical curiosity but rather has interesting applications in many areas of science and engineering such as electrochemistry, control, porous media, electromagnetic, etc. (see [1–3]). There has been a significant development in fractional differential equations in recent years (see [4–16]).

In [14], Zhang and Su have investigated the existence of a solution of the linear fractional differential equation with nonlinear boundary conditions:

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) - du(t) = h(t), \quad t \in (0,T],\ 0 < \alpha < 1, \\ g(u(0)) = u(T), \end{cases}$$

where $d \geq 0$, $h \in C([0,T])$, and ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative. Using the method of upper and lower solutions in reverse order, they get some existence results.

Furthermore, Zhang in [15] has considered the existence of solutions of the following nonlinear fractional differential equation with nonlinear boundary conditions:

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) = f(t, u(t)), \quad t \in (0,T],\ 0 < \alpha < 1, \\ g(u(0), u(T)) = 0, \end{cases}$$

where $f \in C((0,T] \times R, R)$.

Meanwhile, the study of coupled systems involving fractional differential equations is also important as such systems occur in various problems of applied nature; see [4,9]. In [9], Su established sufficient conditions for the existence of solutions

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for a two-point boundary value problem for a coupled system of fractional differential equations:

$$\begin{cases} D_t^p u(t) = f(t, v(t), D_t^q v(t)), & t \in J = [0, 1], \\ D_t^u v(t) = g(t, u(t), D_t^v u(t)), & t \in J = [0, 1], \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

where D_t denotes the Riemann–Liouville fractional derivative, $1 < p, u < 2, q, v > 0, p - v \geq 1, u - q \geq 1$ and $f, g : [0, 1] \times R^2 \longrightarrow R$ are given functions. Ahmad and Nieto [4], extended the results of [9], to a three-point boundary value problem for the following coupled system of fractional differential equations:

$$\begin{cases} D_t^p x(t) = f(t, y(t), D_t^q y(t)), & t \in J = [0, 1], \\ D_t^u y(t) = g(t, x(t), D_t^v x(t)), & t \in J = [0, 1], \\ x(0) = 0, \qquad x(1) = \gamma x(\eta), \\ y(0) = 0, \qquad y(1) = \gamma y(\eta), \end{cases}$$

where D_t denotes the Riemann–Liouville fractional derivative, $1 < p, u < 2, q, v, \gamma > 0, 0 < \eta < 1 p - v \geq 1, u - q \geq 1$, $\gamma \eta^{\alpha-1} < 1, \gamma \eta^{\beta-1} < 1$, and $f, g : [0, 1] \times R^2 \longrightarrow R$ are given functions.

However, as far as we know, there are few existence results for fractional differential systems obtained by use of the monotone iterative technique, which is a powerful tool to get the extremal solutions for various integer differential equations. We will consider the following systems of fractional differential equations with nonlinear boundary conditions making use of the monotone iterative technique

$$\begin{cases} {}^cD_{0+}^{\alpha} u(t) = f_1(t, u(t), v(t), {}^cD_{0+}^{p} u(t), {}^cD_{0+}^{q} u(t)), & t \in J = [0, T], \\ {}^cD_{0+}^{\alpha} v(t) = f_2(t, v(t), u(t), {}^cD_{0+}^{p} v(t), {}^cD_{0+}^{q} v(t)), & t \in J = [0, T], \\ g_1(v(0), v(T), u(0), u(T)) = 0, \qquad g_2(v(0), v(T), u(0), u(T)) = 0, \\ u'(0) = b, \qquad v'(0) = d, \end{cases} \tag{1.1}$$

where ${}^cD_{0+}^{\alpha}$ denotes the Caputo fractional derivative, $1 < \alpha \leq 2, p, q > 0, \alpha - p \geq 1, \alpha - q \geq 1, f_1, f_2 \in C(J \times R^4, R)$, $g_1, g_2 \in C(R^4, R), b, d \in R$.

This paper is organized as follows. In Section 2, we establish a new comparison principle and discuss the uniqueness of the generalized solutions to linear problem of (1.1). In Section 3, we obtain the existence of extremal generalized solutions for (1.1) by utilizing the monotone iterative technique and the method of lower and upper generalized solutions.

2. Linear problem and comparison principle

Let us introduce a space: $X = \{u(t) | u(t) \in C^1(J)\}$ endowed with the norm $\|u\| = \max_{t\in J} |u(t)| + \max_{t\in J} |u'(t)|$. Indeed, $(X, \|.\|_X)$ is a Banach space. Obviously, for $(u, v) \in X \times X$, $(X \times X, \|.\|_{X\times X})$ is a Banach space with $\|(u, v)\|_{X\times X} = \|u\|_X + \|v\|_X$.

For the convenience of the readers, we first present some useful definitions and fundamental facts of fractional calculus theory, which can be found in [1,3].

Definition 2.1. Caputo's derivative for a function $f \in C^n([0, \infty), R)$ can be written as

$${}^cD_{0+}^s f(x) = \frac{1}{\Gamma(n-s)} \int_0^x \frac{f^{(n)}(t)dt}{(x-t)^{s+1-n}}, \quad n = [s] + 1 \tag{2.1}$$

where $[s]$ denotes the integer part of real number $s > 0$.

Definition 2.2. For $s > 0$, the integral

$$I_{0+}^s f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}} dt, \tag{2.2}$$

is called the Riemann–Liouville fractional integral of order s.

Definition 2.3. For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^s f(x) = \frac{1}{\Gamma(n-s)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{s-n+1}} dt, \quad n = [s] + 1 \tag{2.3}$$

is called the Riemann–Liouville fractional derivative of order $s > 0$.

Lemma 2.1 ([1,3]). *Let $u \in C^m[0, T]$ and $q \in (m-1, m]$, $m \in N$. Then for $t \in [0, T]$,*

$$I^q({}^cD^q_{0+}u(t)) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0). \tag{2.4}$$

Now let us consider a linear initial value problem, which is important for us to obtain the existence of generalized solutions for problem (1.1).

Lemma 2.2. *Let $M, N, N_1, a, b \in R$, $\varphi(t) \in C(J)$. If $u(t) \in C^2[0, T]$ is a solution of the following initial value problems:*

$$\begin{cases} {}^cD^\alpha_{0+}u(t) - Mu(t) - N{}^cD^p_{0+}u(t) - N_1{}^cD^q_{0+}u(t) = \varphi(t), \quad t \in J, \\ u(0) = a, \\ u'(0) = b, \end{cases} \tag{2.5}$$

where $1 < \alpha \le 2$, $p, q > 0$, $\alpha - p \ge 1$, $\alpha - q \ge 1$, then $u(t)$ is a solution of the following integral equation:

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(\varphi(s) + Mu(s))ds + \frac{N}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1}u(s)ds \\ & - \frac{aNt^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1}u(s)ds - \frac{aN_1t^{\alpha-q}}{\Gamma(\alpha-q+1)} + a + bt. \end{aligned} \tag{2.6}$$

Proof. By (2.4), for some constants c_1, c_2, we have

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\varphi(s) + Mu(s) + N{}^cD^p_{0+}u(s) + N_1{}^cD^q_{0+}u(s)]ds + c_1 + c_2t \\ = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(\varphi(s) + Mu(s))ds + \frac{N}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1}u(s)ds \\ & - \frac{Nu(0)t^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1}u(s)ds - \frac{N_1u(0)t^{\alpha-q}}{\Gamma(\alpha-q+1)} + c_1 + c_2t. \end{aligned}$$

Hence,

$$\begin{aligned} u'(t) = {} & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}(\varphi(s) + Mu(s))ds + \frac{N}{\Gamma(\alpha-p-1)} \int_0^t (t-s)^{\alpha-p-2}u(s)ds \\ & - \frac{Nu(0)t^{\alpha-p-1}}{\Gamma(\alpha-p)} + \frac{N_1}{\Gamma(\alpha-q-1)} \int_0^t (t-s)^{\alpha-q-2}u(s)ds - \frac{N_1u(0)t^{\alpha-q-1}}{\Gamma(\alpha-q)} + c_2. \end{aligned}$$

Applying the initial value conditions $u(0) = a$, $u'(0) = b$, we get $c_1 = a$, $c_2 = b$.

Consequently, we obtain

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\varphi(s) + Mu(s) + N{}^cD^p_{0+}u(s) + N_1{}^cD^q_{0+}u(s)]ds + a + bt \\ = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(\varphi(s) + Mu(s))ds + \frac{N}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1}u(s)ds \\ & - \frac{aNt^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1}u(s)ds - \frac{aN_1t^{\alpha-q}}{\Gamma(\alpha-q+1)} + a + bt. \quad \square \end{aligned} \tag{2.7}$$

Definition 2.4. We say that $u(t) \in C^1(J, R)$ is a generalized solution of the initial value problem (2.5) if $u(t)$ is a solution of the integral equation (2.6).

Remark. In fact, if $u(t) \in C^2(J, R)$ is a solution of (2.5), we easily get $u(t) \in C^1(J, R)$ is a generalized solution of (2.5) from Lemma 2.2. However, by the following simple example, a generalized solution of (2.5) is not a solution of (2.5) in general.

Example. In (2.5), we suppose $\varphi(t) = 1$, $M = N = N_1 = 0$, $T = 1$, $\alpha = \frac{3}{2}$. According to (2.7), we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t (t-s)^{\frac{1}{2}} ds + a + bt \\ &= \frac{2}{3\Gamma\left(\frac{3}{2}\right)} t^{\frac{3}{2}} + a + bt. \end{aligned}$$

Hence, $u'(t) = \frac{1}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} + b$, which implies $u(t) \notin C^2[0, 1]$. According to the definition of Caputo derivative, we could not define Caputo derivative ${}^cD_{0+}^{\alpha} u(t)$.

Lemma 2.3. *Suppose that $M, N, N_1 \geq 0$ and the following inequality*

$$\frac{MT^{\alpha}}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1 T^{\alpha-q}}{\Gamma(\alpha-q+1)} < 1, \tag{2.8}$$

holds. If $u \in X$ such that

$$\begin{cases} u(t) \leq \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} [Mu(s) + N {}^cD_{0+}^{p} u(s) + N_1 {}^cD_{0+}^{q} u(s)] ds + u(0) + u'(0)t, \\ u(0) \leq 0, \\ u'(0) \leq 0, \end{cases} \tag{2.9}$$

then $u(t) \leq 0$ for all $t \in J$.

Proof. Suppose that the inequality $u(t) \leq 0, t \in J$ is not true. It means that there exists a $t^* \in J$ such that $u(t^*) > 0$. Let $u(t^*) = \max\{u(t) : t \in J\} = \rho$, $\rho > 0$, then we obtain that

$$\begin{aligned} u(t) &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mu(s) + N {}^cD_{0+}^{p} u(s) + N_1 {}^cD_{0+}^{q} u(s)] ds + u(0) + u'(0)t \\ &= \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + \frac{N}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1} u(s) ds - \frac{N u(0) t^{\alpha-p}}{\Gamma(\alpha-p+1)} \\ &\quad + \frac{N_1}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1} u(s) ds - \frac{N_1 u(0) t^{\alpha-q}}{\Gamma(\alpha-q+1)} + u(0) + u'(0)t \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + \frac{N}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1} u(s) ds + \frac{N_1}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1} u(s) ds \\ &\leq \rho \left(\frac{MT^{\alpha}}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1 T^{\alpha-q}}{\Gamma(\alpha-q+1)} \right). \end{aligned}$$

Let $t = t^*$, we have

$$\rho \leq \rho \left(\frac{MT^{\alpha}}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1 T^{\alpha-q}}{\Gamma(\alpha-q+1)} \right).$$

So

$$\frac{MT^{\alpha}}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1 T^{\alpha-q}}{\Gamma(\alpha-q+1)} \geq 1,$$

which contradicts (2.8). Hence $u(t) \leq 0$ for all $t \in J$. The proof is complete. □

Lemma 2.4. *Assume that $M \geq M_1 \geq 0$, $N, N_1 \geq 0$ and*

$$\frac{(M+M_1)T^{\alpha}}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1 T^{\alpha-q}}{\Gamma(\alpha-q+1)} < 1. \tag{2.10}$$

If $(u, v) \in X \times X$ *such that*

$$\begin{cases} u(t) \le \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[Mu(s) + M_1 v(s) + N^c D_{0+}^p u(s) + N_1{}^c D_{0+}^q u(s)]ds + u(0) + u'(0)t, & t \in J \\ v(t) \le \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[Mv(s) + M_1 u(s) + N^c D_{0+}^p v(s) + N_1{}^c D_{0+}^q v(s)]ds + v(0) + v'(0)t, & t \in J \\ u(0) \le 0, \quad v(0) \le 0, \\ u'(0) \le 0, \quad v'(0) \le 0, \end{cases} \tag{2.11}$$

then $u(t) \le 0$, $v(t) \le 0$, $t \in J$.

Proof. Put $w(t) = u(t) + v(t)$, $\forall t \in J$. Then by (2.11) we have

$$\begin{cases} w(t) \le \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[(M+M_1)w(s) + N^c D_{0+}^p w(s) + N_1{}^c D_{0+}^q w(s)]ds + w(0) + w'(0)t, & t \in J \\ w(0) \le 0, \\ w'(0) \le 0. \end{cases} \tag{2.12}$$

Thus, by (2.10) and Lemma 2.3, we have $w(t) \le 0$, $\forall t \in J$.

That is

$$u(t) + v(t) \le 0. \tag{2.13}$$

By (2.11) and (2.13), we infer that both u and v satisfy the following inequalities:

$$\begin{cases} z(t) \le \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[(M-M_1)z(s) + N^c D_{0+}^p z(s) + N_1{}^c D_{0+}^q z(s)]ds + z(0) + z'(0)t, & t \in J \\ z(0) \le 0, \\ z'(0) \le 0. \end{cases}$$

Lemma 2.3 implies that $u(t) \le 0$, $v(t) \le 0$, $\forall t \in J$. The proof is complete. □

Consider the following problem:

$$\begin{cases} {}^c D_{0+}^\alpha u(t) - Mu(t) - M_1 v(t) - N^c D_{0+}^p u(t) - N_1{}^c D_{0+}^q u(t) = \varphi(t), & t \in J, \ 1 < \alpha \le 2, \\ {}^c D_{0+}^\alpha v(t) - Mv(t) - M_1 u(t) - N^c D_{0+}^p v(t) - N_1{}^c D_{0+}^q v(t) = \psi(t), & t \in J, \\ u(0) = a, \qquad u'(0) = b, \\ v(0) = c, \qquad v'(0) = d, \end{cases} \tag{2.14}$$

where $a, b, c, d \in R$.

Similarly, if $(u, v) \in X \times X$ such that

$$\begin{cases} u(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[\varphi(s) + Mu(s) + M_1 v(s) + N^c D_{0+}^p u(s) + N_1{}^c D_{0+}^q u(s)]ds + a + bt, & t \in J, \\ v(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[\psi(s) + Mv(s) + M_1 u(s) + N^c D_{0+}^p v(s) + N_1{}^c D_{0+}^q v(s)]ds + c + dt, & t \in J, \end{cases}$$

then we call (u, v) a generalized solution of (2.14).

Lemma 2.5. *Assume that* $M, M_1, N, N_1 \ge 0$ *are constants and the following inequality holds*

$$(\alpha + T)\frac{(M+M_1)T^{\alpha-1}}{\Gamma(\alpha+1)} + (\alpha - p + T)\frac{NT^{\alpha-p-1}}{\Gamma(\alpha-p+1)} + (\alpha - q + T)\frac{N_1 T^{\alpha-q-1}}{\Gamma(\alpha-q+1)} < 1, \tag{2.15}$$

then (2.14) *has a unique generalized solution.*

Proof. We first define an operator $T : X \times X \to X \times X$ by

$$\begin{aligned} T(u, v)(t) &= \begin{pmatrix} T_1(u, v)(t) \\ T_2(u, v)(t) \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[\varphi(s) + Mu(s) + M_1 v(s) + N^c D_{0+}^p u(s) + N_1{}^c D_{0+}^q u(s)]ds + a + bt, \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[\psi(s) + Mv(s) + M_1 u(s) + N^c D_{0+}^p v(s) + N_1{}^c D_{0+}^q v(s)]ds + c + dt, \end{pmatrix}. \end{aligned}$$

Now for $(u_2, v_2), (u_1, v_1) \in X \times X$ and for any $t \in J$, we get

$$\begin{aligned}
|T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1}[M(u_2(s) - u_1(s)) + M_1(v_2(s) - v_1(s))\right.\\
&\quad \left. + N({}^cD^p_{0+}u_2(s) - {}^cD^p_{0+}u_1(s)) + N_1({}^cD^q_{0+}u_2(s) - {}^cD^q_{0+}u_1(s))]\right| ds\\
&\le \frac{MT^\alpha}{\Gamma(\alpha+1)}\|u_2 - u_1\| + \frac{M_1T^\alpha}{\Gamma(\alpha+1)}\|v_2 - v_1\|\\
&\quad + \left(N\frac{T^{\alpha-p}}{\Gamma(\alpha-p+1)} + N_1\frac{T^{\alpha-q}}{\Gamma(\alpha-q+1)}\right)\|u_2 - u_1\|\\
&= \left[\frac{MT^\alpha}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{N_1T^{\alpha-q}}{\Gamma(\alpha-q+1)}\right]\|u_2 - u_1\|\\
&\quad + \frac{M_1T^\alpha}{\Gamma(\alpha+1)}\|v_2 - v_1\|,\\
|(T_1(u_2, v_2))'(t) - (T_1(u_1, v_1))'(t)| &= \frac{1}{\Gamma(\alpha-1)}\left|\int_0^t (t-s)^{\alpha-2}[M(u_2(s) - u_1(s)) + M_1(v_2(s) - v_1(s))\right.\\
&\quad \left. + N({}^cD^p_{0+}u_2(s) - {}^cD^p_{0+}u_1(s)) + N_1({}^cD^q_{0+}u_2(s) - {}^cD^q_{0+}u_1(s))]\right| ds\\
&\le \left[\frac{MT^{\alpha-1}}{\Gamma(\alpha)} + \frac{NT^{\alpha-p-1}}{\Gamma(\alpha-p)} + \frac{N_1T^{\alpha-q-1}}{\Gamma(\alpha-q)}\right]\|u_2 - u_1\| + \frac{M_1T^{\alpha-1}}{\Gamma(\alpha)}\|v_2 - v_1\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)\|\\
&\quad \le \left[\left(\frac{MT^{\alpha-1}(\alpha+T)}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p-1}(\alpha-p+T)}{\Gamma(\alpha-p+1)} + \frac{N_1T^{\alpha-q-1}(\alpha-q+T)}{\Gamma(\alpha-q+1)}\right)\right]\|u_2 - u_1\|\\
&\qquad + (\alpha+T)\frac{M_1T^{\alpha-1}}{\Gamma(\alpha+1)}\|v_2 - v_1\|.
\end{aligned}$$

Similar to the above discussion, we can obtain

$$\begin{aligned}
&\|T_2(u_2, v_2)(t) - T_2(u_1, v_1)(t)\|\\
&\quad \le \left[\left(\frac{MT^{\alpha-1}(\alpha+T)}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p-1}(\alpha-p+T)}{\Gamma(\alpha-p+1)} + \frac{N_1T^{\alpha-q-1}(\alpha-q+T)}{\Gamma(\alpha-q+1)}\right)\right]\|v_2 - v_1\|\\
&\qquad + (\alpha+T)\frac{M_1T^{\alpha-1}}{\Gamma(\alpha+1)}\|u_2 - u_1\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|T(u_2, v_2)(t) - T(u_1, v_1)(t)\| &= \|T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)\| + \|T_2(u_2, v_2)(t) - T_2(u_1, v_1)(t)\|\\
&\le \left[\frac{(M+M_1)T^{\alpha-1}(\alpha+T)}{\Gamma(\alpha+1)} + \frac{NT^{\alpha-p-1}(\alpha-p+T)}{\Gamma(\alpha-p+1)} + \frac{N_1T^{\alpha-q-1}(\alpha-q+T)}{\Gamma(\alpha-q+1)}\right]\\
&\quad \times (\|u_2 - u_1\| + \|v_2 - v_1\|).
\end{aligned}$$

By (2.15) we know T is a contraction operator. Consequently, by Banach fixed point theorem, (2.14) has a unique generalized solution $(u, v) \in X \times X$. The proof is complete. □

3. Main results

In this section, we mainly prove the existence of extremal generalized solutions of problem (1.1) by the method of lower and upper generalized solutions and the monotone iterative technique.

Definition 3.1. We say that $(\underline{u_0}, \underline{v_0}) \in X \times X$ is a lower generalized solution of (1.1) if

$$\begin{cases} \underline{u_0}(t) \le \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f_1(s, \underline{u_0}(s), \underline{v_0}(s), {}^cD_{0+}^p \underline{u_0}(s), {}^cD_{0+}^q \underline{u_0}(s))ds + \underline{u_0}(0) + \underline{u_0'}(0)t, & t \in J, \\ \underline{v_0}(t) \le \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f_2(s, \underline{v_0}(s), \underline{u_0}(s), {}^cD_{0+}^p \underline{v_0}(s), {}^cD_{0+}^q \underline{v_0}(s))ds + \underline{v_0}(0) + \underline{v_0'}(0)t, & t \in J, \\ g_1(\underline{v_0}(0), \underline{v_0}(T), \underline{u_0}(0), \underline{u_0}(T)) \le 0, \\ g_2(\underline{v_0}(0), \underline{v_0}(T), \underline{u_0}(0), \underline{u_0}(T)) \le 0, \\ \underline{u_0'}(0) \le b, \\ \underline{v_0'}(0) \le d. \end{cases}$$

Analogously, $(\overline{u_0}, \overline{v_0})$ is called an upper generalized solution of (1.1), if the above inequalities are reversed.

We need the following assumptions.

(H1) $(\underline{u_0}, \underline{v_0}), (\overline{u_0}, \overline{v_0})$ are lower and upper generalized solutions of (1.1), respectively, such that $\underline{u_0}(t) \le \overline{u_0}(t), \underline{v_0}(t) \le \overline{v_0}(t)$ on $t \in J$.

(H2) There exist constants M, M_1, N, N_1 where $M \ge M_1 \ge 0$, $N, N_1 \ge 0$, such that

$$\begin{cases} f_1(t, u_1, u_2, u_3, u_4) - f_1(t, v_1, v_2, v_3, v_4) \ge M(u_1 - v_1) + M_1(u_2 - v_2) + N(u_3 - v_3) + N_1(u_4 - v_4), \\ f_2(t, u_1, u_2, u_3, u_4) - f_2(t, v_1, v_2, v_3, v_4) \ge M(u_1 - v_1) + M_1(u_2 - v_2) + N(u_3 - v_3) + N_1(u_4 - v_4), \end{cases}$$

where $\underline{u_0}(t) \le v_1 \le u_1 \le \overline{u_0}(t), \underline{v_0}(t) \le v_2 \le u_2 \le \overline{v_0}(t), {}^cD_{0+}^p \underline{u_0}(t) \le v_3 \le u_3 \le {}^cD_{0+}^p \overline{u_0}(t), {}^cD_{0+}^q \underline{u_0}(t) \le v_4 \le u_4 \le {}^cD_{0+}^q \overline{u_0}(t), \forall t \in J$.

(H3) There exist constants $r_1, r_2, r_4, r_2', r_3', r_4' \ge 0$ such that

$$g_1(w_1, w_2, w_3, w_4) - g_1(z_1, z_2, z_3, z_4) \le \sum_{i=1}^{2} r_i(z_i - w_i) + (w_3 - z_3) + r_4(z_4 - w_4),$$

$$g_2(w_1, w_2, w_3, w_4) - g_2(z_1, z_2, z_3, z_4) \le (w_1 - z_1) + \sum_{i=2}^{4} r_i'(z_i - w_i),$$

where $\underline{v_0}(0) \le z_1 \le w_1 \le \overline{v_0}(0), \underline{v_0}(T) \le z_2 \le w_2 \le \overline{v_0}(T), \underline{u_0}(0) \le z_3 \le w_3 \le \overline{u_0}(0), \underline{u_0}(T) \le z_4 \le w_4 \le \overline{u_0}(T)$.

In the following, for any $w, z \in X$ with $w(t) \le z(t), \forall t \in J$, we define the ordered interval $[w, z] = \{u \in X : w(t) \le u(t) \le z(t), t \in J\}$.

Theorem 3.1. *Suppose that conditions* (H1)–(H3) *and* (2.15) *hold. Then* (1.1) *has two extremal generalized solutions* $(\overline{u}, \overline{v}), (\underline{u}, \underline{v}) \in [\underline{u_0}, \overline{u_0}] \times [\underline{v_0}, \overline{v_0}]$. *Moreover, there exist monotone iterative sequences* $\{\overline{u_n}\}, \{\underline{u_n}\} \subset [\underline{u_0}, \overline{u_0}], \{\overline{v_n}\}, \{\underline{v_n}\} \subset [\underline{v_0}, \overline{v_0}]$ *such that* $(\overline{u_n}, \overline{v_n}) \to (\overline{u}, \overline{v}), (\underline{u_n}, \underline{v_n}) \to (\underline{u}, \underline{v}) (n \to \infty)$ *uniformly on* $t \in J$, *and*

$$\begin{aligned} &\underline{u_0} \le \underline{u_1} \le \cdots \underline{u_n} \le \cdots \le \underline{u} \le \overline{u} \le \cdots \le \overline{u_n} \le \cdots \le \overline{u_1} \le \overline{u_0}. \\ &\underline{v_0} \le \underline{v_1} \le \cdots \le \underline{v_n} \le \cdots \le \underline{v} \le \overline{v} \le \cdots \le \overline{v_n} \le \cdots \le \overline{v_1} \le \overline{v_0}. \end{aligned} \tag{3.1}$$

Proof. $\forall u_{n-1}, v_{n-1} \in X$, we consider

$$\begin{cases} {}^cD_{0+}^\alpha u_n(t) - Mu_n(t) - M_1 v_n(t) - N{}^cD_{0+}^p u_n(t) - N_1{}^cD_{0+}^q u_n(t) \\ \quad = f_1(t, u_{n-1}(t), v_{n-1}(t), {}^cD_{0+}^p u_{n-1}(t), {}^cD_{0+}^q u_{n-1}(t)) - Mu_{n-1}(t) - M_1 v_{n-1}(t) \\ \qquad - N{}^cD_{0+}^p u_{n-1}(t) - N_1{}^cD_{0+}^q u_{n-1}(t), \quad t \in J, \\ {}^cD_{0+}^\alpha v_n(t) - Mv_n(t) - M_1 u_n(t) - N{}^cD_{0+}^p v_n(t) - N_1{}^cD_{0+}^q v_n(t) \\ \quad = f_2(t, v_{n-1}(t), u_{n-1}(t), {}^cD_{0+}^p v_{n-1}(t), {}^cD_{0+}^q v_{n-1}(t)) - Mv_{n-1}(t) - M_1 u_{n-1}(t) \\ \qquad - N{}^cD_{0+}^p v_{n-1}(t) - N_1{}^cD_{0+}^q v_{n-1}(t), \quad t \in J, \\ u_n(0) = u_{n-1}(0) - g_1(v_{n-1}(0), v_{n-1}(T), u_{n-1}(0), u_{n-1}(T)), \\ v_n(0) = v_{n-1}(0) - g_2(v_{n-1}(0), v_{n-1}(T), u_{n-1}(0), u_{n-1}(T)), \\ u_n'(0) = b, \qquad v_n'(0) = d, \end{cases}$$

which has the following generalized solutions from Lemma 2.5

$$\begin{cases} u_n(t)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}[f_1(s,u_{n-1}(s),v_{n-1}(s),{}^cD^p_{0+}u_{n-1}(s),{}^cD^q_{0+}u_{n-1}(s))-M(u_{n-1}(s)-u_n(s)) \\ \quad -M_1(v_{n-1}(s)-v_n(s))-N({}^cD^p_{0+}u_{n-1}(s)-{}^cD^p_{0+}u_n(s))-N_1({}^cD^q_{0+}u_{n-1}(s)-{}^cD^q_{0+}u_n(s))]ds \\ \quad +u_n(0)+u_n'(0)t, \\ v_n(t)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}[f_2(s,v_{n-1}(s),u_{n-1}(s),{}^cD^p_{0+}v_{n-1}(s),{}^cD^q_{0+}v_{n-1}(s))-M(v_{n-1}(s)-v_n(s)) \\ \quad -M_1(u_{n-1}(s)-u_n(s))-N({}^cD^p_{0+}v_{n-1}(s)-{}^cD^p_{0+}v_n(s))-N_1({}^cD^q_{0+}v_{n-1}(s)-{}^cD^q_{0+}v_n(s))]ds \\ \quad +v_n(0)+v_n'(0)t. \end{cases} \tag{3.2}$$

In this way, we may define the sequences$\{\overline{u_n}(t)\}$, $\{\overline{v_n}(t)\}$ and $\{\underline{u_n}(t)\}$, $\{\underline{v_n}(t)\}$.

Now, we verify that $\{\overline{u_n}(t)\}$, $\{\overline{v_n}(t)\}$ satisfy $\overline{u_n}\le\overline{u_{n-1}}$, $\overline{v_n}\le\overline{v_{n-1}}$, $n=1,2,\ldots$.

Let $w=\overline{u_1}-\overline{u_0}$, $z=\overline{v_1}-\overline{v_0}$. We have

$$\begin{cases} w(t)\le\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}[Mw(s)+M_1z(s)+N^cD^p_{0+}w(s)+N_1{}^cD^q_{0+}w(s)]ds+w(0)+w'(0)t, \quad t\in J, \\ z(t)\le\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}[Mz(s)+M_1w(s)+N^cD^p_{0+}z(s)+N_1{}^cD^q_{0+}z(s)]ds+z(0)+z'(0)t, \quad t\in J, \\ w(0)=\overline{u_1}(0)-\overline{u_0}(0)=-g_1(\overline{v_0}(0),\overline{v_0}(T),\overline{u_0}(0),\overline{u_0}(T))\le 0, \\ z(0)=\overline{v_1}(0)-\overline{v_0}(0)=-g_2(\overline{v_0}(0),\overline{v_0}(T),\overline{u_0}(0),\overline{u_0}(T))\le 0, \\ w'(0)=\overline{u_1'}(0)-\overline{u_0'}(0)\le 0, \\ z'(0)=\overline{v_1'}(0)-\overline{v_0'}(0)\le 0. \end{cases}$$

By Lemma 2.4, we have that $w(t)\le 0$, $z(t)\le 0$, $\forall t\in J$.

Suppose that $\overline{u_i}\le\overline{u_{i-1}}$, $\overline{v_i}\le\overline{v_{i-1}}$, for some $i\ge 1$.

When $n=i+1$,let $w=\overline{u_{i+1}}-\overline{u_i}$, $z=\overline{v_{i+1}}-\overline{v_i}$. We have

$$\begin{aligned} w(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_1(s,\overline{u_i}(s),\overline{v_i}(s),{}^cD^p_{0+}\overline{u_i}(s),{}^cD^q_{0+}\overline{u_i}(s)) \\ &\quad -f_1(s,\overline{u_{i-1}}(s),\overline{v_{i-1}}(s),{}^cD^p_{0+}\overline{u_{i-1}}(s),{}^cD^q_{0+}\overline{u_{i-1}}(s))+M(\overline{u_{i+1}}(s)-\overline{u_i}(s)) \\ &\quad -M(\overline{u_i}(s)-\overline{u_{i-1}}(s))+M_1(\overline{v_{i+1}}(s)-\overline{v_i}(s))-M_1(\overline{v_i}(s)-\overline{v_{i-1}}(s)) \\ &\quad +N({}^cD^p_{0+}\overline{u_{i+1}}(s)-{}^cD^p_{0+}\overline{u_i}(s))-N({}^cD^p_{0+}\overline{u_i}(s)-{}^cD^p_{0+}\overline{u_{i-1}}(s))+N_1({}^cD^q_{0+}\overline{u_{i+1}}(s)-{}^cD^q_{0+}\overline{u_i}(s)) \\ &\quad -N_1({}^cD^q_{0+}\overline{u_i}(s)-{}^cD^q_{0+}\overline{u_{i-1}}(s))]ds+\overline{u_{i+1}}(0)-\overline{u_i}(0)+[\overline{u_{i+1}'}(0)-\overline{u_i'}(0)]t \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Mw(s)+M_1z(s)+N^cD^p_{0+}w(s)+N_1{}^cD^q_{0+}w(s)]ds+w(0)+w'(0)t. \end{aligned}$$

Analogously, we obtain

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f_2(s,\overline{v_i}(s),\overline{u_i}(s),{}^cD^p_{0+}\overline{v_i}(s),{}^cD^q_{0+}\overline{v_i}(s)) \\ &\quad -f_2(s,\overline{v_{i-1}}(s),\overline{u_{i-1}}(s),{}^cD^p_{0+}\overline{v_{i-1}}(s),{}^cD^q_{0+}\overline{v_{i-1}}(s))+M(\overline{v_{i+1}}(s)-\overline{v_i}(s)) \\ &\quad -M(\overline{v_i}(s)-\overline{v_{i-1}}(s))+M_1(\overline{u_{i+1}}(s)-\overline{u_i}(s))-M_1(\overline{u_i}(s)-\overline{u_{i-1}}(s)) \\ &\quad +N({}^cD^p_{0+}\overline{v_{i+1}}(s)-{}^cD^p_{0+}\overline{v_i}(s))-N({}^cD^p_{0+}\overline{v_i}(s)-{}^cD^p_{0+}\overline{v_{i-1}}(s))+N_1({}^cD^q_{0+}\overline{v_{i+1}}(s)-{}^cD^q_{0+}\overline{v_i}(s)) \\ &\quad -N_1({}^cD^q_{0+}\overline{v_i}(s)-{}^cD^q_{0+}\overline{v_{i-1}}(s))]ds+\overline{v_{i+1}}(0)-\overline{v_i}(0)+[\overline{v_{i+1}'}(0)-\overline{v_i'}(0)]t \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Mz(s)+M_1w(s)+N^cD^p_{0+}z(s)+N_1{}^cD^q_{0+}z(s)]ds+z(0)+z'(0)t. \end{aligned}$$

Besides,

$$\begin{aligned} w(0) &= \overline{u_{i+1}}(0)-\overline{u_i}(0) \\ &= \overline{u_i}(0)-\overline{u_{i-1}}(0)-g_1(\overline{v_i}(0),\overline{v_i}(T),\overline{u_i}(0),\overline{u_i}(T))+g_1(\overline{v_{i-1}}(0),\overline{v_{i-1}}(T),\overline{u_{i-1}}(0),\overline{u_{i-1}}(T)) \\ &\le r_1(\overline{v_i}(0)-\overline{v_{i-1}}(0))+r_2(\overline{v_i}(T)-\overline{v_{i-1}}(T))+r_4(\overline{u_i}(T)-\overline{u_{i-1}}(T)) \\ &\le 0, \end{aligned}$$

$$\begin{aligned} z(0) &= \overline{v_{i+1}}(0) - \overline{v_i}(0) \\ &= \overline{v_i}(0) - \overline{v_{i-1}}(0) - g_2(\overline{v_i}(0), \overline{v_i}(T), \overline{u_i}(0), \overline{u_i}(T)) + g_2(\overline{v_{i-1}}(0), \overline{v_{i-1}}(T), \overline{u_{i-1}}(0), \overline{u_{i-1}}(T)) \\ &\le r_2'(\overline{v_i}(T) - \overline{v_{i-1}}(T)) + r_3'(\overline{u_i}(0) - \overline{u_{i-1}}(0)) + r_4'(\overline{u_i}(T) - \overline{u_{i-1}}(T)) \\ &\le 0. \end{aligned}$$

Moreover, $w'(0) = 0, z'(0) = 0$.

By Lemma 2.4, we may obtain $w(t) \le 0, z(t) \le 0$, for $t \in J$. Hence $\overline{u_{i+1}} \le \overline{u_i}$, $\overline{v_{i+1}} \le \overline{v_i}$.

By induction, we have

$$\overline{u_n} \le \cdots \le \overline{u_1} \le \overline{u_0}, \qquad \overline{v_n} \le \cdots \le \overline{v_1} \le \overline{v_0}.$$

Similarly, we may obtain

$$\begin{aligned} &\underline{u_0} \le \underline{u_1} \le \cdots \le \underline{u_n}, \qquad \underline{v_0} \le \underline{v_1} \le \cdots \le \underline{v_n}, \\ &\underline{u_n} \le \overline{u_n}, \qquad \underline{v_n} \le \overline{v_n} \quad n = 0, 1, 2, \ldots. \end{aligned}$$

From the above discussion, we can conclude that

$$\begin{aligned} &\underline{u_0} \le \underline{u_1} \le \cdots \le \underline{u_n} \le \cdots \le \overline{u_n} \le \cdots \le \overline{u_1} \le \overline{u_0}, \\ &\underline{v_0} \le \underline{v_1} \le \cdots \le \underline{v_n} \le \cdots \le \overline{v_n} \le \cdots \le \overline{v_1} \le \overline{v_0}. \end{aligned} \tag{3.3}$$

It is not difficult to show that there exist $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v})$ such that $\lim_{n\to\infty}(\overline{u_n}, \overline{v_n}) = (\overline{u}, \overline{v})$, $\lim_{n\to\infty}(\underline{u_n}, \underline{v_n}) = (\underline{u}, \underline{v})$ uniformly on J. Moreover, $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v})$ are generalized solutions of (1.1) in $[\underline{u_0}, \overline{u_0}] \times [\underline{v_0}, \overline{v_0}]$, and (3.1) is true.

Finally, we prove that $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v})$ are two extremal generalized solutions. Assume that $(u, v) \in [\underline{u_0}, \overline{u_0}] \times [\underline{v_0}, \overline{v_0}]$ is any generalized solution of (1.1).

Suppose that there exists positive integer i such that $\underline{u_i}(t) \le u(t) \le \overline{u_i}(t)$, $\underline{v_i}(t) \le v(t) \le \overline{v_i}(t)$ on J.

Let $w(t) = u(t) - \overline{u_{i+1}}(t)$, $z(t) = v(t) - \overline{v_{i+1}}(t)$, we have

$$\begin{aligned} w(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f_1(s, u(s), v(s), {}^cD_{0+}^p u(s), {}^cD_{0+}^q u(s)) \\ &\quad - f_1(s, \overline{u_i}(s), \overline{v_i}(s), {}^cD_{0+}^p \overline{u_i}(s), {}^cD_{0+}^q \overline{u_i}(s)) - M(\overline{u_{i+1}}(s) - \overline{u_i}(s)) \\ &\quad - M_1(\overline{v_{i+1}}(s) - \overline{v_i}(s)) - N({}^cD_{0+}^p \overline{u_{i+1}}(s) - {}^cD_{0+}^p \overline{u_i}(s)) \\ &\quad - N_1({}^cD_{0+}^q \overline{u_{i+1}}(s) - {}^cD_{0+}^q \overline{u_i}(s))]ds + u(0) - \overline{u_{i+1}}(0) + [u'(0) - \overline{u_{i+1}'}(0)]t \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mw(s) + M_1 z(s) + N{}^cD_{0+}^p w(s) + N_1{}^cD_{0+}^q w(s)]ds + w(0) + w'(0)t. \end{aligned}$$

Analogously, we have

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f_2(s, v(s), u(s), {}^cD_{0+}^p v(s), {}^cD_{0+}^q v(s)) \\ &\quad - f_2(s, \overline{v_i}(s), \overline{u_i}(s), {}^cD_{0+}^p \overline{v_i}(s), {}^cD_{0+}^q \overline{v_i}(s)) - M(\overline{v_{i+1}}(s) - \overline{v_i}(s)) \\ &\quad - M_1(\overline{u_{i+1}}(s) - \overline{u_i}(s)) - N({}^cD_{0+}^p \overline{v_{i+1}}(s) - {}^cD_{0+}^p \overline{v_i}(s)) \\ &\quad - N_1({}^cD_{0+}^q \overline{v_{i+1}}(s) - {}^cD_{0+}^q \overline{v_i}(s))]ds + v(0) - \overline{v_{i+1}}(0) + [v'(0) - \overline{v_{i+1}'}(0)]t \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mz(s) + M_1 w(s) + N{}^cD_{0+}^p z(s) + N_1{}^cD_{0+}^q z(s)]ds + z(0) + z'(0)t. \end{aligned}$$

Besides,

$$\begin{aligned} w(0) &= u(0) - \overline{u_{i+1}}(0) \\ &= u(0) - \overline{u_i}(0) - g_1(v(0), v(T), u(0), u(T)) + g_1(\overline{v_i}(0), \overline{v_i}(T), \overline{u_i}(0), \overline{u_i}(T)) \\ &\le r_1(v(0) - \overline{v_i}(0)) + r_2(v(T) - \overline{v_i}(T)) + r_4(u(T) - \overline{u_i}(T)) \\ &\le 0, \end{aligned}$$

$$\begin{aligned} z(0) &= v(0) - \overline{v_{i+1}}(0) \\ &= v(0) - \overline{v_i}(0) - g_2(v(0), v(T), u(0), u(T)) + g_2(\overline{v_i}(0), \overline{v_i}(T), \overline{u_i}(0), \overline{u_i}(T)) \\ &\le r_2'(v(T) - \overline{v_i}(T)) + r_3'(u(0) - \overline{u_i}(0)) + r_4'(u(T) - \overline{u_i}(T)) \\ &\le 0. \end{aligned}$$

Moreover, $w'(0) = 0, z'(0) = 0$.

By Lemma 2.4, we can obtain $w(t) \le 0, z(t) \le 0$, for $t \in J$.
It means that $u(t) \le \overline{u_{i+1}}(t), v(t) \le \overline{v_{i+1}}(t)$ on J.
Because $u(t) \le \overline{u_0}(t), v(t) \le \overline{v_0}(t)$, on J, by induction, we get that $u(t) \le \overline{u_n}(t), v(t) \le \overline{v_n}(t)$, on J, for every n.
On the analogy of the above way, we also can obtain that $u(t) \ge \underline{u_n}(t), v(t) \ge \underline{v_n}(t)$ on J, for every n. As a result

$$\underline{u_n}(t) \le u(t) \le \overline{u_n}(t), \qquad \underline{v_n}(t) \le v(t) \le \overline{v_n}(t), \quad \forall t \in J. \tag{3.4}$$

By taking the limits in (3.4) as $n \to \infty$, we have $\underline{u}(t) \le u(t) \le \overline{u}(t), \underline{v}(t) \le v(t) \le \overline{v}(t)$, on J. As a result, (1.1) has two extremal generalized solutions $(\overline{u}, \overline{v}), (\underline{u}, \underline{v}) \in [\underline{u_0}, \overline{u_0}] \times [\underline{v_0}, \overline{v_0}]$. The proof is complete. □

If the lower and upper generalized solutions of (1.1) are defined as follows, we can also get the existence of coupled generalized solutions of (1.1).

Definition 3.2. $(\underline{u_0}, \underline{v_0})$, $(\overline{u_0}, \overline{v_0}) \in X \times X$ are the lower and upper generalized solutions of (1.1), respectively, if

$$\begin{cases} \underline{u_0}(t) \le \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f_1(s, \underline{u_0}(s), \overline{v_0}(s), {}^cD_{0+}^p \underline{u_0}(s), {}^cD_{0+}^q \underline{u_0}(s))ds + \underline{u_0}(0) + \underline{u_0'}(0)t, & t \in J, \\ \underline{v_0}(t) \le \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f_2(s, \underline{v_0}(s), \overline{u_0}(s), {}^cD_{0+}^p \underline{v_0}(s), {}^cD_{0+}^q \underline{v_0}(s))ds + \underline{v_0}(0) + \underline{v_0'}(0)t, & t \in J, \\ g_1(\overline{v_0}(0), \overline{v_0}(T), \underline{u_0}(0), \underline{u_0}(T)) \le 0, \\ g_2(\underline{v_0}(0), \underline{v_0}(T), \overline{u_0}(0), \overline{u_0}(T)) \le 0, \\ \underline{u_0'}(0) \le b, \\ \underline{v_0'}(0) \le d, \end{cases}$$

and

$$\begin{cases} \overline{u_0}(t) \ge \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f_1(s, \overline{u_0}(s), \underline{v_0}(s), {}^cD_{0+}^p \overline{u_0}(s), {}^cD_{0+}^q \overline{u_0}(s))ds + \overline{u_0}(0) + \overline{u_0'}(0)t, & t \in J, \\ \overline{v_0}(t) \ge \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f_2(s, \overline{v_0}(s), \underline{u_0}(s), {}^cD_{0+}^p \overline{v_0}(s), {}^cD_{0+}^q \overline{v_0}(s))ds + \overline{v_0}(0) + \overline{v_0'}(0)t, & t \in J, \\ g_1(\underline{v_0}(0), \underline{v_0}(T), \overline{u_0}(0), \overline{u_0}(T)) \ge 0, \\ g_2(\overline{v_0}(0), \overline{v_0}(T), \underline{u_0}(0), \underline{u_0}(T)) \ge 0, \\ \overline{u_0'}(0) \ge b, \\ \overline{v_0'}(0) \ge d. \end{cases}$$

In this case, we need the following assumptions.

(H2′) There exist constants $M, M_1, N, N_1, M \ge M_1 \ge 0, N, N_1 \ge 0$, such that

$$\begin{cases} f_1(t, u_1, u_2, u_3, u_4) - f_1(t, v_1, v_2, v_3, v_4) \ge M(u_1 - v_1) + M_1(v_2 - u_2) + N(u_3 - v_3) + N_1(u_4 - v_4), \\ f_2(t, u_1, u_2, u_3, u_4) - f_2(t, v_1, v_2, v_3, v_4) \ge M(u_1 - v_1) + M_1(v_2 - u_2) + N(u_3 - v_3) + N_1(u_4 - v_4), \end{cases}$$

where $\underline{u_0}(t) \le v_1 \le u_1 \le \overline{u_0}(t), \underline{v_0}(t) \le u_2 \le v_2 \le \overline{v_0}(t), {}^cD_{0+}^p \underline{u_0}(t) \le v_3 \le u_3 \le {}^cD_{0+}^p \overline{u_0}(t), {}^cD_{0+}^q \underline{u_0}(t) \le v_4 \le u_4 \le {}^cD_{0+}^q \overline{u_0}(t), \forall t \in J$.

(H3′) There exist constants $r_1, r_2, r_4, r_2', r_3', r_4' \ge 0$ $(i = 1, 2, 3, 4)$, such that

$$g_1(w_1, w_2, w_3, w_4) - g_1(z_1, z_2, z_3, z_4) \le \sum_{i=1}^{2} r_i(w_i - z_i) + (w_3 - z_3) + r_4(z_4 - w_4),$$

$$g_2(z_1, z_2, z_3, z_4) - g_2(w_1, w_2, w_3, w_4) \le (z_1 - w_1) + r_2'(w_2 - z_2) + \sum_{i=3}^{4} r_i'(z_i - w_i),$$

where $\underline{v_0}(0) \le w_1 \le z_1 \le \overline{v_0}(0), \underline{v_0}(T) \le w_2 \le z_2 \le \overline{v_0}(T), \underline{u_0}(0) \le z_3 \le w_3 \le \overline{u_0}(0), \underline{u_0}(T) \le z_4 \le w_4 \le \overline{u_0}(T)$.

Theorem 3.2. *Suppose that conditions* (H1), (H2′), (H3′) *and* (2.15) *hold. Then* (1.1) *has generalized solutions* $(\overline{u}, \overline{v}), (\underline{u}, \underline{v}) \in [\underline{u_0}, \overline{u_0}] \times [\underline{v_0}, \overline{v_0}]$. *Moreover, there exist monotone iterative sequence* $\{\overline{u_n}\}, \{\underline{u_n}\} \subset [\underline{u_0}, \overline{u_0}], \{\overline{v_n}\}, \{\underline{v_n}\} \subset [\underline{v_0}, \overline{v_0}]$ *such that* $(\overline{u_n}, \underline{v_n}) \to (\overline{u}, \underline{v}), (\underline{u_n}, \overline{v_n}) \to (\underline{u}, \overline{v})(n \to \infty)$ *uniformly on* $t \in J$, *and*

$$\underline{u_0} \le \underline{u_1} \le \cdots \le \underline{u_n} \le \cdots \le \underline{u} \le \overline{u} \le \cdots \le \overline{u_n} \le \cdots \le \overline{u_1} \le \overline{u_0}.$$
$$\underline{v_0} \le \underline{v_1} \le \cdots \le \underline{v_n} \le \cdots \le \underline{v} \le \overline{v} \le \cdots \le \overline{v_n} \le \cdots \le \overline{v_1} \le \overline{v_0}.$$

Proof. $\forall u_{n-1}, v_{n-1} \in X$, we consider

$$\begin{cases} {}^cD_{0+}^{\alpha} u_n(t) - Mu_n(t) + M_1 v_n(t) - N {}^cD_{0+}^{p} u_n(t) - N_1 {}^cD_{0+}^{q} u_n(t) \\ \quad = f_1(t, u_{n-1}(t), v_{n-1}(t), {}^cD_{0+}^{p} u_{n-1}(t), {}^cD_{0+}^{q} u_{n-1}(t)) - Mu_{n-1}(t) + M_1 v_{n-1}(t) - N {}^cD_{0+}^{p} u_{n-1}(t) \\ \qquad - N_1 {}^cD_{0+}^{q} u_{n-1}(t), \quad t \in J, \\ {}^cD_{0+}^{\alpha} v_n(t) - Mv_n(t) + M_1 u_n(t) - N {}^cD_{0+}^{p} v_n(t) - N_1 {}^cD_{0+}^{q} v_n(t) \\ \quad = f_2(t, v_{n-1}(t), u_{n-1}(t), {}^cD_{0+}^{p} v_{n-1}(t), {}^cD_{0+}^{q} v_{n-1}(t)) - Mv_{n-1}(t) + M_1 u_{n-1}(t) \\ \qquad - N {}^cD_{0+}^{p} v_{n-1}(t) - N_1 {}^cD_{0+}^{q} v_{n-1}(t), \quad t \in J, \\ u_n(0) = u_{n-1}(0) - g_1(v_{n-1}(0), v_{n-1}(T), u_{n-1}(0), u_{n-1}(T)), \\ v_n(0) = v_{n-1}(0) - g_2(v_{n-1}(0), v_{n-1}(T), u_{n-1}(0), u_{n-1}(T)), \\ u_n'(0) = b, \qquad v_n'(0) = d, \end{cases}$$

which has the following generalized solutions from Lemma 2.5

$$\begin{cases} u_n(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} [f_1(s, u_{n-1}(s), v_{n-1}(s), {}^cD_{0+}^{p} u_{n-1}(s), {}^cD_{0+}^{q} u_{n-1}(s)) - M(u_{n-1}(s) - u_n(s)) \\ \qquad + M_1(v_{n-1}(s) - v_n(s)) - N({}^cD_{0+}^{p} u_{n-1}(s) - {}^cD_{0+}^{p} u_n(s)) - N_1({}^cD_{0+}^{q} u_{n-1}(s) - {}^cD_{0+}^{q} u_n(s))] ds \\ \qquad + u_n(0) + u_n'(0)t, \\ v_n(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} [f_2(s, v_{n-1}(s), u_{n-1}(s), {}^cD_{0+}^{p} v_{n-1}(s), {}^cD_{0+}^{q} v_{n-1}(s)) - M(v_{n-1}(s) - v_n(s)) \\ \qquad + M_1(u_{n-1}(s) - u_n(s)) - N({}^cD_{0+}^{p} v_{n-1}(s) - {}^cD_{0+}^{p} v_n(s)) - N_1({}^cD_{0+}^{q} v_{n-1}(s) - {}^cD_{0+}^{q} v_n(s))] ds \\ \qquad + v_n(0) + v_n'(0)t. \end{cases}$$

Similar to the proof of Theorem 3.1, we can have the assertion of Theorem 3.2. □

4. Example

Consider the following problems:

$$\begin{cases} {}^cD_{0+}^{\frac{3}{2}} u(t) = \dfrac{t^3+1}{200} u(t) + \dfrac{t^3+1}{200} v(t) + \dfrac{t^4+1}{50} {}^cD_{0+}^{\frac{1}{2}} u(t) + \dfrac{t^4+1}{40} {}^cD_{0+}^{\frac{1}{4}} u(t), \\ {}^cD_{0+}^{\frac{3}{2}} v(t) = \dfrac{t^3+1}{200} v(t) + \dfrac{t^3+1}{200} u(t) + \dfrac{t^4+1}{50} {}^cD_{0+}^{\frac{1}{2}} v(t) + \dfrac{t^4+1}{40} {}^cD_{0+}^{\frac{1}{4}} v(t), \\ -\dfrac{1}{2} v(0) - \dfrac{1}{10} v(1) + u(0) - \dfrac{1}{10} u(1) - 1 = 0, \\ v(0) - \dfrac{1}{10} v(1) - \dfrac{1}{2} u(0) - \dfrac{1}{10} u(1) - 1 = 0, \\ u'(0) = 1, \\ v'(0) = 1, \end{cases} \tag{4.1}$$

where $t \in J = [0, 1]$, $\alpha = \frac{3}{2}$, $p = \frac{1}{2}$, $q = \frac{1}{4}$.

Obviously,

$$\begin{cases} f_1(t, u(t), v(t), {}^cD_{0+}^{\frac{1}{2}} u(t), {}^cD_{0+}^{\frac{1}{4}} u(t)) = \dfrac{t^3+1}{200} u(t) + \dfrac{t^3+1}{200} v(t) + \dfrac{t^4+1}{50} {}^cD_{0+}^{\frac{1}{2}} u(t) + \dfrac{t^4+1}{40} {}^cD_{0+}^{\frac{1}{4}} u(t), \\ f_2(t, v(t), u(t), {}^cD_{0+}^{\frac{1}{2}} v(t), {}^cD_{0+}^{\frac{1}{4}} v(t)) = \dfrac{t^3+1}{200} v(t) + \dfrac{t^3+1}{200} u(t) + \dfrac{t^4+1}{50} {}^cD_{0+}^{\frac{1}{2}} v(t) + \dfrac{t^4+1}{40} {}^cD_{0+}^{\frac{1}{4}} v(t), \\ g_1(v(0), v(1), u(0), u(1)) = -\dfrac{1}{2} v(0) - \dfrac{1}{10} v(1) + u(0) - \dfrac{1}{10} u(1) - 1, \\ g_2(v(0), v(1), u(0), u(1)) = v(0) - \dfrac{1}{10} v(1) - \dfrac{1}{2} u(0) - \dfrac{1}{10} u(1) - 1. \end{cases} \tag{4.2}$$

Taking $(\underline{u_0}(t), \underline{v_0}(t)) = (\frac{t}{10}, \frac{t}{10})$, $(\overline{u_0}(t), \overline{v_0}(t)) = (t^{\frac{3}{2}} + t + 8, t^{\frac{3}{2}} + t + 8)$, then

$$\begin{cases} \underline{u_0}(t) = \dfrac{t}{10} \le \dfrac{1}{\Gamma\left(\frac{3}{2}\right)} \displaystyle\int_0^t (t-s)^{\frac{1}{2}} \left[\frac{s^4+s}{1000} + \frac{s^{\frac{9}{2}} + s^{\frac{1}{2}}}{500\Gamma\left(\frac{3}{2}\right)} + \frac{s^{\frac{19}{4}} + s^{\frac{3}{4}}}{400\Gamma\left(\frac{7}{4}\right)}\right] ds + \dfrac{t}{10}, \\ \underline{v_0}(t) = \dfrac{t}{10} \le \dfrac{1}{\Gamma\left(\frac{3}{2}\right)} \displaystyle\int_0^t (t-s)^{\frac{1}{2}} \left[\frac{s^4+s}{1000} + \frac{s^{\frac{9}{2}} + s^{\frac{1}{2}}}{500\Gamma\left(\frac{3}{2}\right)} + \frac{s^{\frac{19}{4}} + s^{\frac{3}{4}}}{400\Gamma\left(\frac{7}{4}\right)}\right] ds + \dfrac{t}{10}, \\ g_1(\underline{v_0}(0), \underline{v_0}(1), \underline{u_0}(0), \underline{u_0}(1)) = -\dfrac{51}{50} \le 0, \\ g_2(\underline{v_0}(0), \underline{v_0}(1), \underline{u_0}(0), \underline{u_0}(1)) = -\dfrac{51}{50} \le 0, \\ \underline{u_0}'(0) = \dfrac{1}{10} \le 1, \\ \underline{v_0}'(0) = \dfrac{1}{10} \le 1, \end{cases}$$

and

$$\begin{cases} \overline{u_0}(t) = t^{\frac{3}{2}} + t + 8 \\ \quad \ge \dfrac{1}{\Gamma\left(\frac{3}{2}\right)} \displaystyle\int_0^t (t-s)^{\frac{1}{2}} \left[\frac{s^{\frac{9}{2}} + s^4 + s^{\frac{3}{2}} + 8s^3 + s^{\frac{3}{2}} + s + 8}{100} + \frac{\Gamma\left(\frac{5}{2}\right)(s^5+s) + \Gamma\left(\frac{3}{2}\right)\left(s^{\frac{9}{2}} + s^{\frac{1}{2}}\right)}{50} \right. \\ \qquad \left. + \dfrac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{2}\right)\left(s^{\frac{21}{4}} + s^{\frac{5}{4}}\right) + \Gamma\left(\frac{9}{4}\right)\left(s^{\frac{19}{4}} + s^{\frac{3}{4}}\right)}{40\Gamma\left(\frac{9}{4}\right)\Gamma\left(\frac{7}{4}\right)} \right] ds + 8 + t, \\ \overline{v_0}(t) = t^{\frac{3}{2}} + t + 8 \\ \quad \ge \dfrac{1}{\Gamma\left(\frac{3}{2}\right)} \displaystyle\int_0^t (t-s)^{\frac{1}{2}} \left[\frac{s^{\frac{9}{2}} + s^4 + s^{\frac{3}{2}} + 8s^3 + s^{\frac{3}{2}} + s + 8}{100} + \frac{\Gamma\left(\frac{5}{2}\right)(s^5+s) + \Gamma\left(\frac{3}{2}\right)\left(s^{\frac{9}{2}} + s^{\frac{1}{2}}\right)}{50} \right. \\ \qquad \left. + \dfrac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{2}\right)\left(s^{\frac{21}{4}} + s^{\frac{5}{4}}\right) + \Gamma\left(\frac{9}{4}\right)\left(s^{\frac{19}{4}} + s^{\frac{3}{4}}\right)}{40\Gamma\left(\frac{9}{4}\right)\Gamma\left(\frac{7}{4}\right)} \right] ds + 8 + t, \\ g_1(\overline{v_0}(0), \overline{v_0}(1), \overline{u_0}(0), \overline{u_0}(1)) = 1 \le 0, \\ g_2(\overline{v_0}(0), \overline{v_0}(1), \overline{u_0}(0), \overline{u_0}(1)) = 1 \le 0, \\ \overline{u_0}'(0) = 1, \\ \overline{v_0}'(0) = 1, \end{cases}$$

which shows that condition (H1) of Theorem 3.1 holds.

On the other hand, by (4.2), we get that

$$\begin{cases} f_1(t, u_1, u_2, u_3, u_4) - f_1(t, v_1, v_2, v_3, v_4) \ge \dfrac{1}{200}(u_1 - v_1) + \dfrac{1}{200}(v_2 - u_2) + \dfrac{1}{50}(u_3 - v_3) + \dfrac{1}{40}(u_4 - v_4), \\ f_2(t, u_1, u_2, u_3, u_4) - f_2(t, v_1, v_2, v_3, v_4) \ge \dfrac{1}{200}(u_1 - v_1) + \dfrac{1}{200}(v_2 - u_2) + \dfrac{1}{50}(u_3 - v_3) + \dfrac{1}{40}(u_4 - v_4), \end{cases}$$

where $\underline{u_0}(t) \le v_1 \le u_1 \le \overline{u_0}(t)$, $\underline{v_0}(t) \le u_2 \le v_2 \le \overline{v_0}(t)$, ${}^cD_{0+}^p \underline{u_0}(t) \le v_3 \le u_3 \le {}^cD_{0+}^p \overline{u_0}(t)$, ${}^cD_{0+}^q \underline{u_0}(t) \le v_4 \le u_4 \le {}^cD_{0+}^q \overline{u_0}(t)$, $\forall t \in J$.

We see that $M = M_1 = \frac{1}{200}$, $N = \frac{1}{50}$, $N_1 = \frac{1}{40}$ and

$$\begin{aligned} &(\alpha + T)\frac{(M + M_1)T^{\alpha-1}}{\Gamma(\alpha+1)} + (\alpha - p + T)\frac{NT^{\alpha-p-1}}{\Gamma(\alpha-p+1)} + (\alpha - q + T)\frac{N_1T^{\alpha-q-1}}{\Gamma(\alpha-q+1)} \\ &= \frac{5}{2\Gamma\left(\frac{5}{2}\right)} \times \frac{1}{100} + \frac{1}{25} + \frac{9}{160\Gamma\left(\frac{9}{4}\right)} \\ &= 0.1075 < 1. \end{aligned}$$

It is easy to see that (2.15) holds.

Furthermore,

$$g_1(w_1, w_2, w_3, w_4) - g_1(z_1, z_2, z_3, z_4) \geq \frac{1}{4}(z_1 - w_1) + \frac{1}{15}(z_2 - w_2) + (w_3 - z_3) + \frac{1}{20}(z_4 - w_4),$$
$$g_2(w_1, w_2, w_3, w_4) - g_2(z_1, z_2, z_3, z_4) \geq (w_1 - z_1) + \frac{1}{20}(z_2 - w_2) + \frac{1}{8}(z_3 - w_3) + \frac{1}{25}(z_4 - w_4),$$

where $\underline{v_0}(0) \leq z_1 \leq w_1 \leq \overline{v_0}(0)$, $\underline{v_0}(T) \leq z_2 \leq w_2 \leq \overline{v_0}(T)$, $\underline{u_0}(0) \leq z_3 \leq w_3 \leq \overline{u_0}(0)$, $\underline{u_0}(T) \leq z_4 \leq w_4 \leq \overline{u_0}(T)$.

For $r_1 = \frac{1}{4}, r_2 = \frac{1}{15}, r_4 = r_2' = \frac{1}{20}, r_3' = \frac{1}{8}, r_4' = \frac{1}{25}$, it is not difficult to see that condition (H_3) holds.

Thus, all conditions of Theorem 3.1 are satisfied. Therefore, (4.1) has two extremal generalized solutions $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v}) \in [\underline{u_0}, \overline{u_0}] \times [\underline{v_0}, \overline{v_0}]$, which can be obtained by taking limits from some iterative sequences.

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